→ [Momentum] Flux Driven Turbulence

· Turbulent Pipe Flow

(cf. Landau, Lifshitz "Fluid Mechanics")

Till now → homogeneous flow in a periodic box
→ cascade in scale space (Kolmogorov)

Now → inhomogeneous flow in a pipe
→ momentum transport in a turbulent boundry layer (Prandtl)

Consider turbulent pipe flow:

→ core

→ logarithmic profile (inertial sublayer) (virtually all high Re experiments → "profile consistency")

→ linear profile (viscous sublayer)

Common features of pipe flow:

- linear → logarithmic $U(x)$ profile

- logarithmic profile persists over a broad range of Re

$$\left(Re = 2Ua/\nu \right)$$

∴ logarithmic profile "universal" ("Prandtl "Law of the Wall")

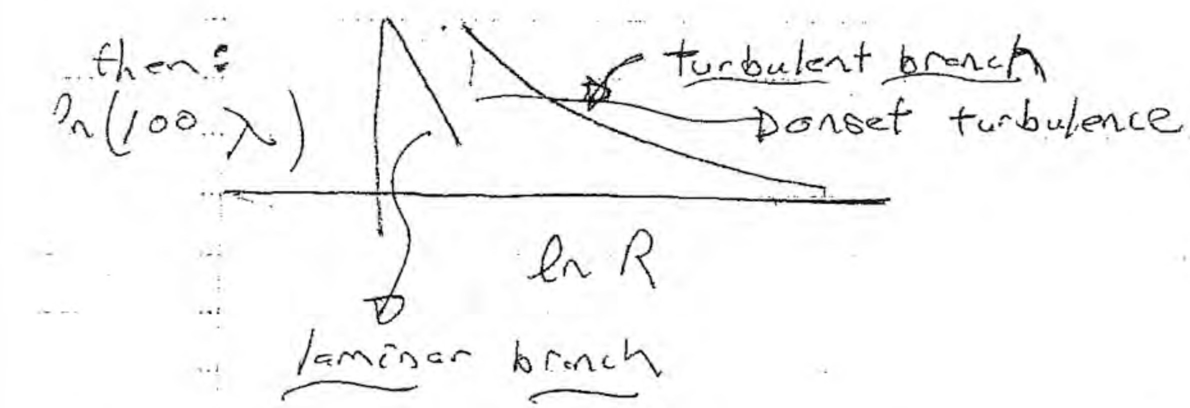
- resistance ~~discontinuously~~ increases with increasing Re,

pressure drop/length

$$\lambda = \frac{2a\,\Delta p/\ell}{\frac{1}{2}\rho U^2}$$

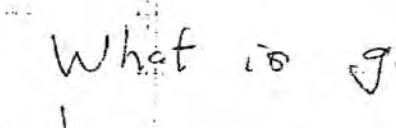
$\frac{1}{2}\rho U^2$ ⟶ mean flow energy

then:



- turbulent resistance curve universal.

What is going on?



no slip boundary condition

$$U = U(x) \xrightarrow[x \to 0]{} 0$$

$$\therefore\ U = U(x) \Rightarrow \left\{ \begin{array}{l} \text{momentum flux} \\ \text{to wall} \end{array} \right.$$

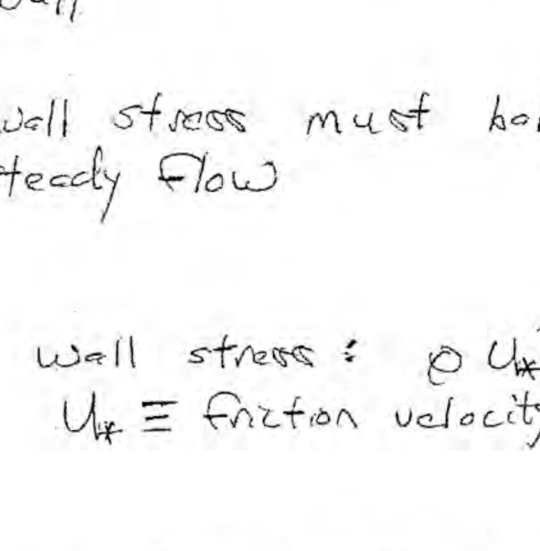
→ momentum flux to wall ⇒ stress on the wall

→ wall stress must balance pressure drop for steady flow

so wall stress : ρU_*^2
$U_* \equiv$ friction velocity

i.e.
$\underline{V}\cdot\nabla\underline{V} - \nabla\cdot\widetilde{v'v'} = -\widetilde{v'v'} \sim \frac{\Delta P}{\ell}$
$F \sim SA$ $\quad F \sim \Delta A$
$2\pi V_*^2 \ell R \sim \Delta P \pi R^2$

∴ $\rho U_*^2 2\pi a \, \ell = \Delta p \, \pi a^2$



Force on wall $\cong$
$\rho U_*^2 A_{wall}$

(Pressure Drop) A_{Flow}

$\equiv$ Force on Fluid

ition only ⇒ $\rho U_*^2 (2\pi a \ell) = (\Delta p) \pi a^2$

$$U_* = \left[(\Delta p / 2\rho) \left(\frac{a}{2\ell} \right) \right]^{1/2}$$

Friction Velocity

U_* ≡ friction velocity
≡ "typical" velocity of turbulence in turbulent pipe

Deriving the inertial sublayer profile:

i) dimensional reasoning

in pipe flow inertial sublayer, have

3 dimensional parameters ρ, τ, x

$\underbrace{\rho}_{\text{density}}$, $\underbrace{\tau}_{\substack{\text{wall stress} \\ U_*}}$, $x \to$ distance from wall

Key Point: Assumption of scale invariance

on scale $l_{vs} = \frac{\nu}{U_*} < x < a$

⇒ universality of logarithmic profile motivates scale invariance assumption

Now, seek velocity gradient dU/dx,

$\frac{dU}{dx}$: U_*, x, ρ

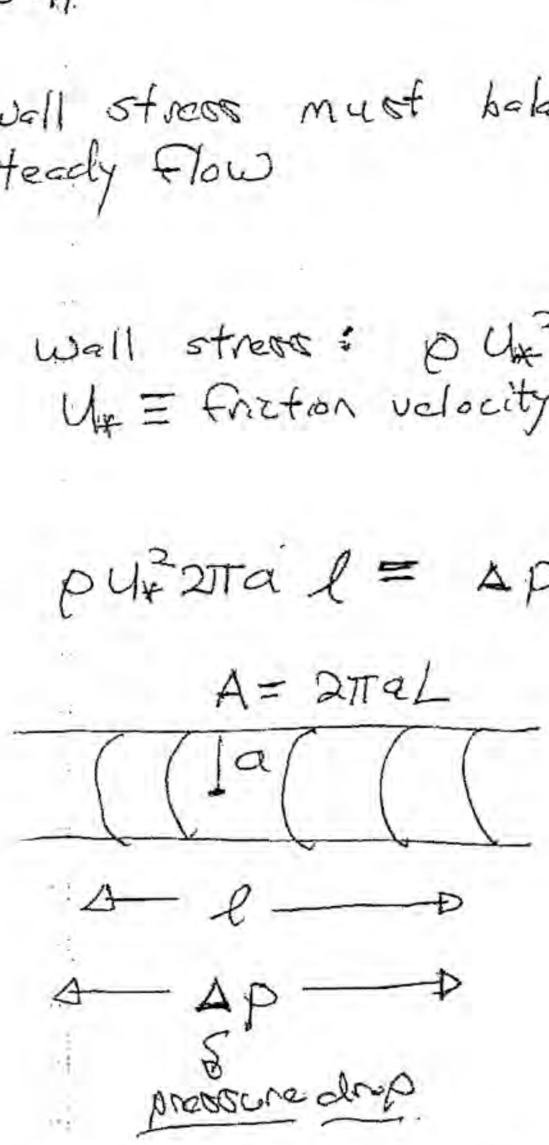
→ Momentum flux to wall ⇒ stress on the wall

→ wall stress must balance pressure drop, for steady flow

so wall stress : ρU_*^2
$U_* \equiv$ friction velocity

i.e.
$$\underbrace{\underline{v}\cdot\nabla\underline{v}}_{F\sim SA} - \nu\nabla^2\underline{v} = \underbrace{-\nabla p}_{F\sim pA} \sim \frac{\Delta p}{\ell}$$
$$2\pi v_*^2 \ell R \sim \Delta p \pi R^2$$

$$\therefore \quad \rho U_*^2 2\pi a \ell = \Delta p \pi a^2$$



Force on wall $\simeq$ $\rho U_*^2 A_{wall}$

(Pressure Drop) A_{Flow} = Force on Fluid

ition only ⇒ $\rho U_*^2 (2\pi a \ell) = (\Delta p)\pi a^2$

$$U_* = \left[(\Delta p / 2\rho)\left(\frac{a}{2\ell}\right)\right]^{1/2}$$

Friction Velocity

so simplest form for dU/dx is:

$$\frac{dU}{dx} = \frac{U_*}{x}$$

$$\Rightarrow \boxed{\begin{aligned} U &= \frac{U_*}{K} \ln(x/x_0) \\ &= \frac{U_*}{K} \ln x \quad + \text{const.} \end{aligned}}$$

→ logarithmic profile (consequence of scale invariance in pipe flow)

→ $K \sim .4$ universal constant → Von-Karman constant

$x_0 \leftrightarrow$ width of viscous sublayer $\sim \nu/U_*$

i.) Physical Reasoning

stationary flow ⇒

momentum flux to wall = pressure drop

$\therefore \quad \underbrace{\langle \tilde{V}_x \tilde{V}_z \rangle}_{\text{Reynolds stress}} = u_*^2$

$$\rho \langle \tilde{V}_x \tilde{V}_z \rangle = \Gamma_p$$

Γ_p → momentum flux

$$\Gamma_p / \rho = u_*^2$$

Now, to calculate $\langle \tilde{V}_x \tilde{V}_z \rangle$:

→ take velocity fluctuation as generated by mixing of $U(x)$, so

$$\rightarrow \quad \tilde{V}_z \sim \underbrace{\ell}_{\text{"mixing length"}} \frac{\partial U}{\partial x}$$

analogous to Chapman-Enskog expansion, i.e.

$\ell \Leftrightarrow \ell_{mfp}$.

$\tilde{V}_x \Leftrightarrow V_{Th}$.

Where scale invariance $\Leftrightarrow$ $\ell \sim x$

mixing length set by distance from wall

so

$$\langle \tilde{V}_x \tilde{V}_z \rangle = \langle V_x \ell \rangle \frac{\partial U}{\partial x}$$

$$\cong U_* x \frac{\partial U}{\partial x}$$

$\nu_T = U_* x \rightarrow$ "eddy viscosity" / "turbulent viscosity" > Key concept.

$\Rightarrow$ rate of turbulent transport of momentum

then momentum balance $\Rightarrow$

$$U_* x \frac{\partial U}{\partial x} = U_*^2$$

$$\Rightarrow \quad U = \frac{U_*}{K} \ln(x/x_0) \rightarrow \text{Logarithmic Profile}$$

$\rightarrow$ $\boxed{\text{Law of the Wall}}$

Some comments:

⇒ as in K41, clear phenomenology critical to guiding the approximations ⟺ scale invariance

≅ "Mixing length theory always works ... provided you know the mixing length..."
— P.D.

→ why a single value of velocity, i.e. U_*?

Consistent with mixing length hypothesis, velocity fluctuations generated by mixing of mean flow gradient, i.e.

⇒

$$\tilde{V} \sim l \frac{\partial U}{\partial x} \sim x \frac{\partial U}{\partial x}$$

$$\sim \cancel{x} \frac{U_*}{\cancel{x}}$$

absence of preferred scale.

consistent .'. Assumption consistent with:
- logarithmic profile
- scale invariance.

⇒ viscous sublayer / cut-off of inertial layer ?

$\therefore$ when : $\nu_T < \nu$ { molecular viscosity dominates mixing

$$\Rightarrow \quad u_* x \lesssim \nu$$

$$x \lesssim \nu / u_* \equiv \underbrace{x_\nu}_{\text{viscous sublayer scale.}}$$

In viscous sublayer, flow linear :

$$\nu \frac{\partial u}{\partial x} = u_*^2$$

$$\therefore \quad u = \frac{u_*^2}{\nu} x$$

⇒ note effect of turbulence is to :

- flatten profile – { higher transport at fixed wall stress
- reduce central velocity
- limit Q (quality factor)

– matching, for const:

$$x_0 = \nu / u_* \quad \text{so}$$

$$u = \frac{u_*}{\kappa} \ln\left(\frac{u_* y}{\nu}\right)$$

Note: Flow in viscous sublayer is turbulent, but mixing there affected by dissipation range scales ⇒ linear profile

Now – turbulent dissipation?

Consider NSE:

$$\frac{\partial \hat{\underline{v}}}{\partial t} + \hat{\underline{v}} \cdot \nabla \hat{\underline{v}} + \langle v_z \rangle \frac{\partial}{\partial z} \hat{\underline{v}} + \hat{v}_x \frac{\partial}{\partial x} \langle v_z \rangle$$

$$= -\nabla \hat{p} + \nu \nabla^2 \hat{\underline{v}}$$

$\hat{\underline{v}} \cdot$ and avg ⇒

$$\frac{\partial}{\partial t} \langle \hat{v}^2 \rangle + \langle \hat{\underline{v}} \cdot \hat{\underline{v}} \cdot \nabla \hat{\underline{v}} \rangle + \langle v_z \rangle \underbrace{\cancel{\left\langle \hat{\underline{v}} \cdot \frac{\partial}{\partial z} \hat{v} \right\rangle}}_{\text{odd}}$$

$$+ \langle \hat{v}_x \hat{v}_z \rangle \frac{\partial}{\partial x} \langle v_z \rangle = -\underbrace{\cancel{\langle \hat{\underline{v}} \cdot \nabla \hat{p} \rangle}}_{\text{i.b.p}} - \nu \langle (\nabla \hat{v})^2 \rangle$$

For net energy budget:

$$\partial_t \mathcal{E} = \underbrace{-\langle \tilde{V}_x \tilde{V}_z \rangle \frac{\partial \langle V_z \rangle}{\partial x}}_{\substack{\text{input to fluctuations} \\ \text{by relaxation of} \\ \text{mean shear flow} \\ \text{(Reynolds work)}}} \underbrace{- \nu \langle (\nabla \tilde{V})^2 \rangle}_{\substack{\text{dissipation} \\ \text{of fluctuation} \\ \text{energy by viscosity}}}$$

$\therefore$ can define:

$$\underbrace{\epsilon}_{\substack{\text{turbulent} \\ \text{dissipation} \\ \text{rate}}} = \langle \tilde{V}_x \tilde{V}_z \rangle \frac{\partial U}{\partial x}$$

and using mixing length theory:

$$\langle \tilde{V}_x \tilde{V}_z \rangle = U_* x \frac{\partial U}{\partial x}$$

$$\Rightarrow \epsilon = (U_* x) \left(\frac{\partial U}{\partial x} \right)^2 = \underbrace{\nu_T \left(\frac{\partial U}{\partial x} \right)^2}_{\substack{\text{rate of "heating" by} \\ \text{turbulent relaxation} \\ \text{of mean flow.}}}$$

input $\to$ mean flow mixing

obviously: $\nu\langle(\nabla\tilde{v})^2\rangle = \nu_T\left(\frac{\partial U}{\partial x}\right)^2$

small scale dissipation

and,

$$\epsilon = (u_* x)\left(\frac{u_*}{x}\right)^2 \qquad (\text{ignoring } R)$$

$$= \frac{u_*^3}{x}$$

$\to$ sets dissipation rate.

i.e. $\epsilon = \frac{v_0^3}{\ell}$ $\qquad V_0 \Leftrightarrow u_*$, $\ell \Leftrightarrow x$

$\to$ ϵ finite as $\nu \to 0$ (i.e. viscous sublayer gradient diverges then)

Additional References:

- S. B. Pope, "Turbulent Flows"

- H. Tennekes and J. Lumley, "A First Course in Turbulence"

→ Now, interesting to tabulate comparison between Pipe Flow and K41 Problem

Pipe Flow (Prandtl)	K41 (Kolmogorov)
scales: a, x, ν/u_*	ℓ_0, ℓ_n, ℓ_d
invariance: $x \to$ real space	$\ell \to$ scale space
inertial sublayer viscous sublayer	inertial range dissipation range
lance: $u_*^2 = \nu_T \frac{\partial u}{\partial x}$	$\epsilon = \frac{v(\ell)^2}{\tau(\ell)}$
Dynamics: eddy viscosity $\nu_T = u_* x$	turn-over rate $1/\tau(\ell) = \frac{v(\ell)}{\ell}$
Result: $u = \frac{u_*}{\kappa} \ln(x)$	$v(\ell) = \epsilon^{1/3} \ell^{1/3}$
Universal profile	universal spectral scaling
Dissipation: $\nu = \nu_T$ $x_0 = \nu/u_*$	$v(\ell)/\ell = \nu/\ell^2$ $\ell_d = \nu^{3/4}/\epsilon^{1/4}$

→ Practical Issues

Resistance Law ⇔ Pipe Flows.

have: $\frac{\nu}{u_*} < x \leq a$
radius

can push to $x \sim a$, with logarithmic accuracy

$$U \cong \frac{u_*}{k} \ln\left(\frac{u_* a}{\nu}\right)$$

but $$v_* = u_* = \left(\frac{a}{\ell}\frac{\Delta P}{2\rho}\right)^{1/2}$$

⇒ can re-write:

$$U = \left(\frac{a\,\Delta P}{2\rho\,\ell k^2}\right)^{1/2} \ln\left(a\left(\frac{a\Delta P}{2\rho\ell}\right)^{1/2}\Big/\nu\right)$$

Convenient to define:

$$\lambda = \frac{2a\overbrace{\Delta P/\ell}^{\text{drive}}}{\underbrace{\tfrac{1}{2}\rho U^2}_{\text{flow KE}}}$$

→ friction factor / resistance coefficient

⇒ taking $R_e = 2aU/\nu$

can re-write friction law as:

$$1/\sqrt{\lambda} = .88 \ln\left(R_e\sqrt{\lambda}\right) - .85$$

phenom.

$R_e = 2aU/\nu$

$$\lambda = \frac{2a\,\Delta p/\ell}{1/2\,\rho U^2}$$

→ good fit to pipe flow data.